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Twisted K-theory of Lie groups

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Abstract: I determine the twisted K-theory of all compact simply connected simple Lie groups. The computation reduces via the Freed-Hopkins-Teleman theorem [1] to the CFT prescription, and thus explains why it gives the correct result. Finally I analyze the exceptions noted by Bouwknegt et al. [2].

Keywords: Topological Field Theories, D-branes, Conformal Field Models in String Theory.

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1. Introduction

WZW models provide a nice playground of solvable conformal field theories. The key to their treatment is of course that the target space is a group manifold G, and everything in the theory should be expressed in terms of the representation theory of this group.

An example for this is the fusion ring of the CFT, which is determined by the representation theory of the loop group LG at the corresponding level. A more sophisticated instance are the Cardy branes, special boundary states in the CFT labeled by the irreducible representations.

In [3, 4] this was combined with boundary RG flow methods to determine possible decay paths of boundary states. The authors then used this to guess the underlying conserved charges and compared it with the expected result, the twisted K-theory of G.

This check was significantly improved in [5], where the predicted charge groups for all $G = \mathrm{SU}(n)$ are compared with a purely K-theoretic computation. However the K-theory computation uses an explicit cell decomposition and thus neither generalizes to other Lie groups nor does it make use of the representation theory. So although one finds at the end that the charge groups match this is not a very satisfactory result.

My work closes this gap: I reduce the computation of the twisted K-theory to a calculation in representation theory, and the latter will be very similar to the CFT calculation. So there will be no mystery that the resulting charge groups match, and furthermore it works for all simply connected, compact, simple Lie groups in the same way. The crucial connection between CFT and K-theory is the theorem of Freed, Hopkins and Teleman [1].

Apart from the still unproven Freed-Hopkins-Teleman result I make use of another conjectural but widely believed piece of information: I assume the existence of fusion potentials (for a more detailed discussion of this technical assumption see section 3.4.1). These two issues are the only possible loopholes in my computation, everything else is known for sure.

The actual computation is closely related to the method in [6] and I would like to thank Sakura Schäfer-Nameki for sharing her draft with me. However the seemingly simpler case of a WZW model instead of a coset is technically quite involved, and this is what I will be dealing with.

During the preparation of this paper a comment of Hopkins was relayed through [7]. Although it suffered from the Chinese whispers phenomenon it clearly suggested to compute the twisted K-theory along the following lines.

2. An example

Take the simplest case, the Lie group $G = \mathrm{SU}(2) \simeq S^3$. The WZW model at level k contains Cardy boundary states $|\lambda\rangle\rangle$, labeled by the first $k+1$ irreps of $\mathrm{SU}(2)$ (or alternatively by irreducible positive energy irreps of $\widetilde{LG}$). Call this index set $\mathcal{J}_k$. Now it was argued in [4] that the charges q_λ of the boundary states satisfy

$$\dim_{\mathbb{C}}(\mu)\, q_\lambda = \sum_{\nu \in \mathcal{J}_k} N_{\mu\lambda}^{\nu}\, q_\nu\,, \tag{2.1}$$

where $N_{\mu\lambda}^{\nu}$ are the fusion coefficients. This has two immediate consequences:

- Take $\lambda = 0$ the trivial representation $\Rightarrow$ $q_\mu = \dim_{\mathbb{C}}(\mu)\, q_0$, every charge is a multiple of q_0.
- Any relation $\sum_i a_i \mu_i = 0$ in the fusion ring leads to

  $$\left(\sum_{i \in \mathcal{J}_k} a_i \dim_{\mathbb{C}}(\mu_i) \right) q_0 = 0 \tag{2.2}$$

  so the charge q_0 is torsion. If there are no further identifications than eq. (2.1) then the order of the torsion is the minimal dimension of a relation in the fusion ring.

Especially for $\mathrm{SU}(2)$ at level k the relations in the fusion ideal have dimensions $(k+2)\mathbb{Z}$, therefore the charge group is $\mathbb{Z}_{k+2}$.

Let us compare this with the twisted K-theory of $\mathrm{SU}(2)$. It is readily evaluated for any given twist class $\tau \in H^3\big(\mathrm{SU}(2)\big) \simeq \mathbb{Z}$, for example Rosenberg's spectral sequence [8] yields immediately that

$$^{\tau}K^*\big(\mathrm{SU}(2)\big) = \mathbb{Z}_\tau\,. \tag{2.3}$$

Although the result nicely matches the CFT if you identify $\tau = k+2$ it does not explain why this should be so. Even from the physicist's perspective this is something of a miracle since the WZW model only has to reproduce the classical geometry in the $k \to \infty$ limit.

I will now explain why the CFT formula works, details of the computation will be explained in the following sections. First introduce G equivariant K-theory (for details, see section 3.1) via

$$ {}^{\tau}K^{*}\left(G\right) = {}^{\tau}K^{*}_{G}\left(G_{\mathrm{Tr}} \times G_{\mathrm{L}}\right) = {}^{\tau}K^{*}_{G}\left(G_{\mathrm{Ad}} \times G_{\mathrm{L}}\right), \tag{2.4} $$

where the subscripts Tr, L, Ad denote the group action on that factor as <u>Tr</u>ivial action, <u>L</u>eft multiplication, <u>Ad</u>joint action. Now the twisted K-theory for a cartesian product can be computed via a Künneth spectral sequence, the result (using FHT, see section 3.3) is that

$$ \begin{aligned} {}^{\tau}K^{*}_{G}\left(G_{\mathrm{Ad}} \times G_{\mathrm{L}}\right) &= \mathrm{Tor}^{*}_{K_G(\{\mathrm{pt.}\})}\Big({}^{\tau}K^{*}_{G}\big(G_{\mathrm{Ad}}\big), K^{*}_{G}\big(G_{\mathrm{L}}\big)\Big) \\ &= \mathrm{Tor}^{*}_{RG}\big(RG/I_k,\, \mathbb{Z}\big) \end{aligned} \tag{2.5} $$

with the Verlinde algebra[1] RG/I_k at level $k = \tau - \check{h}(G)$, where $\check{h}(G)$ is the dual Coxeter number. Especially for $G = \mathrm{SU}(2)$ the representation ring is

$$ RSU(2) = \mathbb{Z}[\Lambda], \tag{2.6} $$

where Λ is the fundamental 2-dimensional representation. In this case the Verlinde algebra is particularly easy to write down. The ideal of relations is generated by $(k+1)$-th symmetric power of Λ, i.e. the irreducible representation in dimension $k+2$. One can get this representation by reducing the $(k+1)$-fold tensor product of Λ:

$$ I_k = \big\langle \mathrm{Sym}_{k+1}\Lambda \big\rangle = \Big\langle \Lambda^{k+1} - \text{lower order terms} \Big\rangle, \tag{2.7} $$

for example $\mathrm{Sym}_2\Lambda = \Lambda^2 - 1$. Now evaluate eq. (2.5). Tor is the derived functor of $\otimes$ which means that we have to do the following:

1. Find a projective resolution of the Verlinde algebra, i.e. a complex whose homology is RG/I_k at position 0 and all entries projective.

2. Apply $- \otimes_{RG} \mathbb{Z}$.

3. Compute the homology of the resulting complex.

Here is a (free hence) projective resolution:

$$ P^{\bullet} \stackrel{\mathrm{def}}{=} \quad 0 \longrightarrow \mathbb{Z}[\Lambda] \xrightarrow{\cdot\left(\mathrm{Sym}_{k+1}\Lambda\right)} \underline{\mathbb{Z}[\Lambda]} \longrightarrow 0 \;, \tag{2.8} $$

[1] *Verlinde algebra* will always denote the algebra over $\mathbb{Z}$, i.e. $RG \simeq \mathbb{Z}\big[x_1, \ldots, x_{\mathrm{rank}(G)}\big]$ modulo the ideal generated by the fusion rules (the *Verlinde ideal* I_k). The associated algebra over $\mathbb{Q}$ will be called *rational Verlinde algebra*, similarly complexified Verlinde algebra. In the language of [9], the Verlinde algebra is the fusion ring and the complexified Verlinde algebra is the fusion rule algebra.

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where the underlined entry is at position 0. For any ring $R \otimes_R \mathbb{Z} = \mathbb{Z}$, the only problem is to find the map induced from multiplying with $\mathrm{Sym}_{k+1}\Lambda$. For that we have to remember that the $\mathbb{Z}[\Lambda]$-module structure on $\mathbb{Z}$ comes from the $K_G(\{\text{pt.}\})$-module structure on $K(\{\text{pt.}\})$, i.e. tensoring a vectorspace with a representation. Since $K(\{\text{pt.}\}) \simeq \mathbb{Z}$ is the dimension of the vector space this is just multiplication with the dimension of the representation. In our case $\dim_{\mathbb{C}}\big(\mathrm{Sym}_{k+1}\Lambda\big) = k+2$, so

$$P^\bullet \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z} = \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{k+2} \underline{\mathbb{Z}} \longrightarrow 0 \tag{2.9}$$

which is exact except in position 0 where the homology is $\mathbb{Z}_{k+2}$.

We have found that

$${}^{k+\check{h}}K^*\big(\mathrm{SU}(2)\big) = {}^{k+2}K^*\big(\mathrm{SU}(2)\big) = \mathrm{Tor}^*_{RSU(2)}\big(RSU(2)/I_k,\, \mathbb{Z}\big) \simeq \mathbb{Z}_{k+2} \tag{2.10}$$

as we already know from eq. (2.3). But now it is clear from eq. (2.9) that we got it as $\mathbb{Z}/\dim(I_k)$, which is precisely what the CFT predicted. Furthermore eqns. (2.4),(2.5) hold for all simply connected, compact, simple Lie groups, and I will use it to determine all K-groups in the following.

3. The general computation

Fix once and for all a simply connected compact simple Lie group G. In the following we will work with topological spaces together with group actions on them and twist classes on them (the category t-G-Top). To facilitate this I will use the following conventions:

For a given $X \in Ob(t\text{-}G\text{-Top})$ let $X^{G\text{-Top}}$ be the underlying G-space and $t_X \in H^3_G\big(X^{G\text{-Top}}\big)$ the twist[2] class. A map $h : X \to Y$ is a G-map $h^{G\text{-Top}} : X^{G\text{-Top}} \to Y^{G\text{-Top}}$ such that $t_Y = h^*\big(t_X\big)$; In general there need neither be a map from nor to a point.

The cartesian product $X \times Y$ is the ordinary product with twist class $t_{X\times Y} = \pi_1^*(t_X) + \pi_2^*(t_Y)$, where $\pi_{1,2}$ is the projection (they are not maps in t-G-Top) to the first and second factor.

The t_X-twisted equivariant K-theory will be denoted ${}^tK_G(X)$.

3.1 The action map

Write G_{L} for G acting on itself by left multiplication and no twist, $t_{G_L} = 0$. Furthermore let G_{Tr} be G acting trivially on itself, but with arbitrary twist class $t_{G_{\mathrm{Tr}}} \in H^3_G(G_{\mathrm{Tr}}) \simeq \mathbb{Z}$.

The all-important observation is the following [6]: For G_{Ad} (G with the adjoint action on itself) one can pick a twist class such that there is an isomorphism

$$f : G_{\mathrm{Ad}} \times G_{\mathrm{L}} \xrightarrow{\sim} G_{\mathrm{Tr}} \times G_{\mathrm{L}}\,. \tag{3.1}$$

This is the map I used previously in eq. (2.4).

[2] Actually the representative of the cohomology class matters here, different choices may give isomorphic but not canonically isomorphic K-groups. I'm disregarding this detail in the following.

On the underlying G-spaces, f is of course the G-diffeomorphism

$$f^{G\text{-Top}} : G_{\mathrm{Ad}}^{G\text{-Top}} \times G_{\mathrm{L}}^{G\text{-Top}} \stackrel{\sim}{\to} G_{\mathrm{Tr}}^{G\text{-Top}} \times G_{\mathrm{L}}^{G\text{-Top}},\ (g,\gamma) \mapsto (\gamma^{-1} g \gamma, \gamma) \tag{3.2}$$

so we can pick a suitable twist class on $G_{\mathrm{Ad}}^{G\text{-Top}} \times G_{\mathrm{L}}^{G\text{-Top}}$, it remains to show that it comes from the projection on the first factor.

So (dropping the superscripts G-Top for readability) consider

$$p : G_{\mathrm{Ad}} \times G_{\mathrm{L}} \to G_{\mathrm{Ad}}\,. \tag{3.3}$$

We want to show that $p^* : H^3_G\big(G_{\mathrm{Ad}}\big) \to H^3_G\big(G_{\mathrm{Ad}} \times G_{\mathrm{L}}\big) \simeq \mathbb{Z}$ is an isomorphism. By the universal coefficient theorem and the Hurewicz isomorphism it suffices to show that

$$p_* : \pi_i \overbrace{\Big(\big(G_{\mathrm{Ad}} \times G_{\mathrm{L}} \times EG\big)/G\Big)}^{\simeq G_{\mathrm{Ad}}} \to \pi_i\Big(\big(G_{\mathrm{Ad}} \times EG\big)/G\Big) \tag{3.4}$$

is an isomorphism for $i = 2, 3$. This follows from the long exact homotopy sequence of the fibration p:

$$\pi_i(G) \to \pi_i(G_{\mathrm{Ad}}) \stackrel{p_*}{\to} \pi_i\Big(\big(G_{\mathrm{Ad}} \times EG\big)/G\Big) \to \overbrace{\pi_{i-1}(G)}^{=0 \text{ for } i=2,3} \to \cdots\,. \tag{3.5}$$

Following the image of the generating $S^3 \subset G$ one sees that the image of the leftmost map is contractible (zero) since the G-action on G_{Ad} had fixed points.

Obviously isomorphic spaces have the same cohomology, thus

$$^tK^*\,(G) = {}^tK^*_G\,(G_{\mathrm{Tr}} \times G_{\mathrm{L}}) = {}^tK^*_G\,(G_{\mathrm{Ad}} \times G_{\mathrm{L}}) \tag{3.6}$$

3.2 Making contact with FHT

In the next section I will discuss how the K-theory of the factors determines the K-theory of $G_{\mathrm{Ad}} \times G_{\mathrm{L}}$, for now I will summarize the K-theory of the factors, and their RG-module structure (i.e. how tensoring with a ordinary G-representation acts). The easy part is

$$K^i_G(G_{\mathrm{L}}) = K^i\big(\{\mathrm{pt.}\}\big) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} & i \text{ even} \end{cases} \tag{3.7}$$

with RG-action

$$\rho \cdot x = \dim_{\mathbb{C}}(\rho) x \qquad \forall \rho \in RG, x \in K^*_G(G_{\mathrm{L}}) \simeq \mathbb{Z}\,. \tag{3.8}$$

The twisted equivariant K-theory of G_{Ad} is much more complicated and given by

Theorem 1 (Freed-Hopkins-Teleman=FHT)

$$^tK_G^{\dim G}(G_{Ad}) = RG/I_k \tag{3.9}$$

is an isomorphism of RG-modules, where the level $k = t_{G_{Ad}} - \check{h}$.

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3.3 The Künneth spectral sequence

A Künneth theorem in general computes the cohomology of a cartesian product $X \times Y$ from the cohomology of X and Y. Now if you are really lucky you find that $h^*(X \times Y) = h^*(X) \otimes h^*(Y)$, for example if h^* is the usual de Rham cohomology. But in general this is far more difficult.

A rather exhaustive account for untwisted equivariant K-theory is given in [10]. One finds a Künneth spectral sequence which — to make things even worse — does not always yield the K-theory of the product. However for sufficiently nice groups (like G compact simply connected Lie groups, as is the case we are interested in) this spectral sequence does compute the desired K-theory of the product. Unfortunately the proof in [10] uses knowledge of the non-equivariant K-theory of G, which is what we are after in the twisted case.

Fortunately the Künneth spectral sequence was later extended in [11] to K-theory for C^*-algebras. This is useful since we can think of twisted K-theory as the K-theory of some (possibly noncommutative) C^*-algebra, see [12], and we get the following

Theorem 2. *Let G be a simply connected compact Lie group, $X \in Ob(t\text{-}G\text{-}Top)$ and $Y \in Ob(G\text{-}Top)$ (i.e. there is no twist on Y). Then there is a spectral sequence with*

$$E_2^{-p,*} = \mathrm{Tor}^p_{RG}\left({}^tK^*_G\left(X\right), K^*_G\left(Y\right)\right) \tag{3.10}$$

converging to ${}^tK_G(X \times Y)$.

In section 3.5 I will determine the E_2 term for the case at hand. Of course we then have to worry about higher differentials and extension ambiguities, but we will see in section 5 that these are all trivial and hence

$${}^tK^*(G_{\mathrm{Ad}} \times G_{\mathrm{L}}) \simeq E_2^{*,*} = \mathrm{Tor}^*(RG/I_k, \mathbb{Z}). \tag{3.11}$$

3.4 Properties of the Verlinde algebra

Of course it is possible to work out the Verlinde algebra for any given group and level. However to actually compute the Tor in general it would be very helpful if RG/I_k were a complete intersection, that is I_k generated by a regular sequence (this will be defined in the following).

Although it seems to be true, there is no general proof so far. I will show that it would be a consequence of the existence of fusion potentials.

3.4.1 Fusion potentials

A fusion potential is a polynomial

$$\phi \in RG \otimes \mathbb{Q} = \mathbb{Q}\left[x_1, \ldots, x_n\right], \qquad n = \mathrm{rank}\, G \tag{3.12}$$

such that the Verlinde ideal $I_k \subset RG$ is generated by the partial derivatives:

$$I_k = \left\langle \frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n} \right\rangle_{RG}. \tag{3.13}$$

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Such potentials have been determined for the $A_n = \mathrm{SU}(n+1)$ and $C_n = Sp(2n)$ Lie groups for all k. Gepner [13] conjectured that such a potential exists in general, but I do not know of any proof so far. In the following I will assume this to be true.

Actually I will only be using that the Verlinde ideal is generated by n elements. I do not know any proof for this weaker statement either.

3.4.2 Regular sequences

First let us recall the following

Definition 1. *Let R be a ring, then an ordered sequence of elements $y_1, y_2, \dots, y_n \in R$ is a* regular sequence *on R if*

1. *They do not generate the whole ring: $\langle y_1, \dots, y_n \rangle \neq R$*
2. *For $i = 1, \dots, n$, y_i is a nonzerodivisor in $R/\langle y_1, \dots, y_{i-1}\rangle$.*

In general an ideal will have different sets of generators, some might form a regular sequence and some will not. So to make everything explicit (for $R = RG$) one would have to write down generators of the Verlinde ideal for each Lie group G and each level k, and then show that the chosen generators in the chosen ordering form a regular sequence.

Instead, I will take a more high-powered approach and only show the existence of a regular sequence which generates the Verlinde ideal. This is a rather elementary application of the theory of commutative rings.

First we note that the representation ring of G is a polynomial ring generated by the $n \stackrel{\text{def}}{=} \mathrm{rank}(G)$ fundamental representations, $RG \simeq \mathbb{Z}\big[x_1, \dots, x_n\big]$. Since it is a polynomial ring over $\mathbb{Z}$ it is Cohen-Macaulay. This means that for each ideal $I \in RG$ we have[3] $\mathrm{codim}(I) = \mathrm{depth}(I)$. The same holds for the rational representation ring $RG \otimes \mathbb{Q}$.

To simplify notation I will use a subscript $\mathbb{Q}$ to denote the change of base ring from $\mathbb{Z}$ to $\mathbb{Q}$, so if RG/I is the Verlinde algebra then $RG_{\mathbb{Q}}/I_{\mathbb{Q}}$ is the rational Verlinde algebra (I will suppress the level to avoid double subscripts).

The second ingredient is the fact that we are dealing with a RCFT, so the Verlinde algebra is a finite rank torsionfree $\mathbb{Z}$-module, i.e. $RG/I \simeq \mathbb{Z}^d$ for some $d \in \mathbb{Z}_>$.

Especially $\dim\big(RG_{\mathbb{Q}}/I_{\mathbb{Q}}\big) = 0$: From the point of view of algebraic geometry, the spectrum of the rational Verlinde algebra is just a finite set of points. Put differently, the minimal primes over $I_{\mathbb{Q}}$ are maximal ideals.

Recall that the codimension of $I_{\mathbb{Q}}$ is the minimum over all prime ideals $\mathfrak{p} \supset I_{\mathbb{Q}}$ over the maximum of lengths of chains of prime ideals descending from $\mathfrak{p}$. If $\mathfrak{p}$ is maximal then those chains have maximal length and we get $\mathrm{codim}\, I_{\mathbb{Q}} = \dim\, RG_{\mathbb{Q}} = n$, the rank of G.

Now we actually want the codimension of I, and not its rational version $I_{\mathbb{Q}}$. A fancy way of going from $\mathbb{Z}$ to the quotient field $\mathbb{Q}$ is localization at the nonzero integers. The advantage of this description is that it yields a nice relation between the prime ideals in the corresponding polynomial rings, and we get (see e.g. Theorem 36 of [14])

$$\mathrm{codim}\ I = \mathrm{codim}\ I_{\mathbb{Q}} = n\,. \tag{3.14}$$

[3]Here (Co)Dimension always refers to the Krull dimension, i.e. lengths of chains of prime ideals. The Cohen-Macaulay property means that this definition retains some of the intuitive properties of "dimension".

The last piece of information we need is the only point specific to WZW models: the Verlinde ideal I can be generated by n elements, assuming that there is a fusion potential.

By the above remarks we know that $n = \operatorname{depth} I$, so we can apply [14, Theorem 125]: If an ideal I can be generated by $n = \operatorname{depth} I$ elements then it can be generated by a regular sequence (of n elements).

Theorem 3. *Let G be a compact simply connected simple Lie group. Assuming the existence of a fusion potential for the Verlinde algebra of G there exist a regular sequence $y_1, \ldots, y_n \in RG$ of length $n = \operatorname{rank}(G)$ that generates the Verlinde ideal: $\langle y_1, \ldots, y_n\rangle = I_k$.*

3.4.3 Koszul resolutions

Given any sequence $y_1, \ldots, y_n \in RG$ the *Koszul complex* $\mathcal{K}(y_1, \ldots, y_n)$ is a complex of length $n+1$ with the i^{th} entry the degree i piece of the exterior algebra $\bigwedge_n RG$ (see [15] for a nice introduction). Another useful way of thinking about the Koszul complex would be the following: First for one element, $\mathcal{K}(y_1)$ is the length two complex analogous to the one depicted in eq. (2.8). Then $\mathcal{K}(y_1, \ldots, y_{i+1}) = \mathcal{K}(y_1, \ldots, y_i) \otimes \mathcal{K}(y_n)$ (of course now most subtleties are hidden in the tensor product of complexes).

The whole point of this construction is that if $y_1, \ldots, y_n$ form a regular sequence in RG, then $\mathcal{K}(y_1, \ldots, y_n)$ is a resolution of the quotient ring $RG/\langle y_1, \ldots, y_n\rangle$. Especially we can choose (by the preceding section) a regular sequence generating the Verlinde ideal and thus obtain a resolution of the Verlinde algebra.

3.5 Deriving tensor

Finally we have everything to compute the E_2 term of the spectral sequence in theorem 2. We know a projective resolution of the Verlinde algebra, and now it is a rather simple algebraic task to compute the Tor.

Using the Koszul resolution we readily compute

$$\begin{aligned}
\operatorname{Tor}^*_{RG}\left(RG/I_k,\ \mathbb{Z}\right) &= \operatorname{Tor}^*_{RG}\left(RG/\left\langle y_1, \ldots, y_n\right\rangle,\ \mathbb{Z}\right) = \\
&= H_*\left(\mathcal{K}(y_1, \ldots y_n) \otimes_{RG} \mathbb{Z}\right) \\
&= H_*\left(\left(\bigotimes_{i=1}^n \mathcal{K}(y_i)\right) \otimes_{RG} \mathbb{Z}\right) \\
&= H_*\left(\bigotimes_{i=1}^n \left(\mathcal{K}(y_i) \otimes_{RG} \mathbb{Z}\right)\right) \\
&= H_*\left(\bigotimes_{i=1}^n \left[0 \to \mathbb{Z} \xrightarrow{d_i} \underline{\mathbb{Z}} \to 0\right]\right),
\end{aligned} \tag{3.15}$$

where multiplication by $d_i = \dim(y_i)$ is the map induced by y_i as explained in section 2.

The homology of each factor is of course

$$H_r\left(\left[0 \to \mathbb{Z} \xrightarrow{d_i} \underline{\mathbb{Z}} \to 0\right]\right) = \begin{cases} \mathbb{Z}_{d_i} & r = 0 \\ 0 & \text{else} \end{cases}, \tag{3.16}$$

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G	y	r_0	r_1	r	$\check{h}$
$A_1 = \mathrm{SU}(2)$	1	1	0	1	2
A_n, $n \geq 2$ $[\mathrm{SU}(n+1)]$	$\mathrm{lcm}(1,2,\ldots,n)$	2^{n-2}	2^{n-2}	2^{n-1}	$n+1$
B_n, $n \geq 2$	$\mathrm{lcm}(1,2,\ldots,2n-1)$	2^{n-2}	2^{n-2}	2^{n-1}	$2n-1$
C_n, $n \geq 3$	$\mathrm{lcm}(1,2,\ldots,n,$ $,1,3,5,\ldots,2n-1)$	2^{n-2}	2^{n-2}	2^{n-1}	$n+1$
D_n, $n \geq 4$	$\mathrm{lcm}(1,2,\ldots,2n-3)$	2^{n-2}	2^{n-2}	2^{n-1}	$2n-2$
G_2	$\mathrm{lcm}(1,2,\ldots,5) = 60$	1	1	2	4
F_4	$\mathrm{lcm}(1,2,\ldots,11) = 27720$	2^2	2^2	2^3	9
E_6	$\mathrm{lcm}(1,2,\ldots,11) = 27720$	2^4	2^4	2^5	12
E_7	$\mathrm{lcm}(1,2,\ldots,17) =$ $= 12252240$	2^5	2^5	2^6	18
E_8	$\mathrm{lcm}(1,2,\ldots,29) =$ $= 2329089562800$	2^6	2^6	2^7	30

Table 1: Coefficients determining the twisted K-groups, see eq. (3.19)

and all that remains is to apply the usual Künneth formula for chain complexes, the result is

$$H\left(\bigotimes_{i=1}^{n}\left[0 \to \mathbb{Z} \xrightarrow{d_i} \underline{\mathbb{Z}} \to 0\right]\right) = \bigoplus_{j=1}^{2^{n-1}} \mathbb{Z}_{\gcd(d_1,\ldots,d_n)} \tag{3.17}$$

Proof. Induction: It is true for $n = 1$ and

$$\begin{aligned} H_*&\left(\bigotimes_{i=1}^{n+1}\left[0 \to \mathbb{Z} \xrightarrow{d_i} \underline{\mathbb{Z}} \to 0\right]\right) = \\ &= \bigoplus_{j=1}^{2^{n-1}}\left(\mathbb{Z}_{\gcd(d_1,\ldots,d_n)} \otimes \mathbb{Z}_{d_{n+1}}\right) \oplus \bigoplus_{j=1}^{2^{n-1}} \mathrm{Tor}_{\mathbb{Z}}\left(\mathbb{Z}_{\gcd(d_1,\ldots,d_n)}, \mathbb{Z}_{d_{n+1}}\right) = \\ &= \left(\bigoplus_{j=1}^{2^{n-1}} \mathbb{Z}_{\gcd(d_1,\ldots,d_{n+1})}\right) \oplus \left(\bigoplus_{j=1}^{2^{n-1}} \mathbb{Z}_{\gcd(d_1,\ldots,d_{n+1})}\right) \end{aligned} \tag{3.18}$$

Finally note that $\gcd(d_1,\ldots,d_n)$ does not depend on the choice of generators since it is the generator of the image $\dim(I_k) \subset \mathbb{Z}$ under the dimension homomorphism $\dim : RG \to \mathbb{Z}$. Explicitly computing $\gcd(d_1,\ldots,d_n)$ in any special case is straightforward but tedious. Fortunately general expressions were determined in [2], and I will use their results (although they were unable to prove every formula in the cases where no fusion potential is known).

Combining eqns. (3.6), (3.11) and (3.17) we determined ${}^tK^*(G)$. One can also determine the contributions to ${}^tK^{0,1}$ separately by keeping track of even/odd degree terms in Tor^*_{RG}. This is in principle clear but gets somewhat messy to write down, so I avoided it so far. The final result is that

$${}^tK^{\dim G}(G) \simeq (\mathbb{Z}_x)^{r_0}\ , \quad {}^tK^{1+\dim G}(G) \simeq (\mathbb{Z}_x)^{r_1}\ , \quad {}^tK^*(G) \simeq (\mathbb{Z}_x)^{r} \tag{3.19}$$

where

$$x \overset{\text{def}}{=} \frac{t_G}{\gcd(t_G, y)}, \qquad t_G = k + \check{h}(G) \tag{3.20}$$

with all numerical coefficients determined by table 1.

4. There are no exceptions

For certain Lie groups and low levels ($k = 1$ or 2) it was noted in [2] that some of the fundamental representations are zero in the Verlinde algebra. So they proposed that the corresponding generator should be removed, and the CFT rule for computing the D-brane charges should be applied to this presentation of the algebra. Of course this then depends on the explicit presentation of the Verlinde algebra, i.e. the choice of generators and relations.

However (although far from obvious) in the K-theory computation above there is no ambiguity since Tor is independent of the chosen resolution. There is only a technical problem of computing the RG-module action of the Verlinde Algebra in the exceptional cases of [2], which I want to comment on.

So, for example, consider G_2 at level $k = 1$ as in [2]. The Verlinde algebra is

$$V_k(G_2) \simeq \mathbb{Z}[x_1, x_2]\Big/\big\langle x_1,\ x_2^2 - x_1 - x_2 - 1\big\rangle \tag{4.1a}$$

$$= \mathbb{Z}[x_2]\Big/\big\langle x_2^2 - x_2 - 1\big\rangle, \tag{4.1b}$$

where x_1 and x_2 are the two fundamental representations of dimensions 14 and 7.

The generators for the relations in eq. (4.1a) are a regular sequence in RG_2 and we obviously recover the result of table 1 if we compute ${}^tK(G_2)$ that way.

But we are certainly allowed to use the simpler presentation eq. (4.1b) to compute E_2 in the Künneth spectral sequence. We find that

$$\begin{aligned} E_2^{-p,*} &= \operatorname{Tor}_{RG_2}\Big(\mathbb{Z}[x_2]\Big/\big\langle x_2^2 - x_2 - 1\big\rangle,\ \mathbb{Z}\Big) \\ &= H_{-p}\bigg(0 \longrightarrow \mathbb{Z}[x_2] \otimes_{RG_2} \mathbb{Z} \xrightarrow{\ 41\ } \underline{\mathbb{Z}[x_2] \otimes_{RG_2} \mathbb{Z}} \longrightarrow 0\bigg). \end{aligned} \tag{4.2}$$

But now $\mathbb{Z}[x_2] \otimes_{RG_2} \mathbb{Z} \simeq \mathbb{Z}_7$ since

$$7\,(q \otimes n) = q \otimes (7n) = (x_1 q) \otimes n = 0 \otimes n = 0. \tag{4.3}$$

Since 41 is invertible in $\mathbb{Z}_7$ we get

$$E_2^{-p,*} = H_{-p}\bigg(0 \longrightarrow \mathbb{Z}_7 \xrightarrow{\ 41\ } \underline{\mathbb{Z}_7} \longrightarrow 0\bigg) = 0 \quad \forall p \tag{4.4}$$

and thus we reproduce the result of table 1.

This fits together nicely with the physical RG flow picture. Recall that the flows that lead to eq. (2.1) come from perturbations

$$S \ \to\ S + \operatorname{Tr} \mathbf{P} \exp\left(\oint_{\partial\Sigma} J\right) \tag{4.5}$$

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by the holonomy of the spin current $J = \Lambda_\mu J^\mu$ in some representation. Here you may choose any representation of G, regardless of what its image in the Verlinde algebra would be. We have to take all possible flows into account when we determine the D-brane charge group, i.e. we would get the wrong answer if we artificially restricted ourselves to representations which would be nonzero in the Verlinde algebra.

5. An algebra of BPS states

It remains to show that the Künneth spectral sequence for ${}^tK^*_G\big(G_{\mathrm{Ad}} \times G_{\mathrm{L}}\big)$ does actually collapse at the 2^{nd} term, and that we can solve the extension problems. Both can be dealt with by considering multiplicative structures.[4]

Of course twisted K-theory does not come with a multiplication: The twist class adds when you tensor twisted bundles. Instead I will work in K-homology and use the fact that all spaces we consider are actually groups, so they come with multiplication maps $\mu : X \times X \to X$. Push forward induces the Pontryagin product

$$ {}^tK^G_*(X) \otimes_{RG} {}^tK^G_*(X) \longrightarrow {}^tK^G_*(X \times X) \xrightarrow{\mu_*} {}^tK^G_*(X)\,. \tag{5.1} $$

For ${}^tK^G_*(G_{\mathrm{Ad}})$ it was noted in [17] that the Pontryagin product is simply the fusion product in the Verlinde algebra. Since it is easy to identify the Pontryagin product for the trivial group $\{\text{pt.}\}$ we see that

$$ \begin{aligned} {}^tK^G_*(G_{\mathrm{Ad}}) &= RG/I_k \\ K^G_*(G_{\mathrm{L}}) &= K_*\left(\{\text{pt.}\}\right) \simeq \mathbb{Z} \\ K^G_*\big(\{\text{pt.}\}\big) &= RG \end{aligned} \tag{5.2} $$

as rings.

There is a dual version of the Künneth spectral sequence (the bar spectral sequence of [18]) with

$$ E^2_{p,*} = \mathrm{Tor}^{RG}_p\left({}^tK^G_*(X),\ {}^tK^G_*(Y)\right) \quad \Rightarrow\ {}^tK^G_*\left(X \times Y\right) \tag{5.3} $$

so as before we use the trick with the action map of section 3.1 to get a spectral sequence

$$ E^2_{p,*} = \mathrm{Tor}^{RG}_p\left(RG/I_k,\ \mathbb{Z}\right) \quad \Rightarrow\ {}^tK^G_*\left(G_{\mathrm{Ad}} \times G_{\mathrm{L}}\right) = {}^tK_*(G)\,. \tag{5.4} $$

The advantage here is that it is a spectral sequence of RG-algebras.

Lets have a closer look at the E^2 term. The $\mathrm{Tor}(\cdot,\cdot)$ of two algebras is again an algebra: Again I will use the Koszul resolution of the Verlinde algebra (see section 3.4.3), which can be described as the exterior algebra on $n = \mathrm{rank}(G)$ generators of degree 1.

$$ \mathcal{K}(y_1, \dots, y_n) = \Lambda_{RG}(\eta_1, \dots, \eta_n)\,. \tag{5.5} $$

The Verlinde algebra is then the homology of this exterior algebra with respect to the differential $d(\eta_i) = y_i$. Applying $- \otimes \mathbb{Z}$ and taking the homology with respect to $d(\eta_i) = \dim_{\mathbb{C}}(y_i)$ we determine the algebra structure of the E^2 term as

$$ \mathrm{Tor}^{RG}_p\left(RG/I_k,\ \mathbb{Z}\right) = \Lambda_{\mathbb{Z}_x}(\eta_1, \dots, \eta_n)\,. \tag{5.6} $$

[4]Of course this has nothing to do with the algebra of BPS states as in [16].

So the E^2 term looks like this

$$E^2_{p,*} = \begin{tikzcd} 0 & \mathbb{Z}_x & \bigoplus\limits_{i=1..n} \eta_i \mathbb{Z}_x \arrow[ll, bend right, "d_2"'] & \bigoplus\limits_{1\leq i<j\leq n} \eta_i \wedge \eta_j \mathbb{Z}_x \arrow[ll, bend right, "d_2"'] \\ p=-1 & p=0 & p=1 & p=2 \end{tikzcd} \tag{5.7}$$

and d_2 obviously vanishes on $E^2_{1,*}$. Since all the algebra generators are at degree 1, the differential d_2 vanishes identically. By the same reason all higher differentials are zero, and $E^2 = E^\infty$.

It remains to see that ${}^tK_*(G)$ is only x-torsion, in principle there could be nontrivial extensions $0 \to \mathbb{Z}_x \to \mathbb{Z}_{x^2} \to \mathbb{Z}_x \to 0$ when we try to recover the homology from the associated graded groups. The E^∞ term of the spectral sequence are the successive quotients $E^\infty_{p,*} = F^p/F^{p-1}$ of a filtration

$$0 \subset \cdots \subset F^{p-1} \subset F^p \subset F^{p+1} \subset \cdots \subset {}^tK_*(G)\,. \tag{5.8}$$

The first quotient is $\mathbb{Z}_x = E^\infty_{0,*} = F^0/0 = F^0$, and from the multiplicative structure of the exterior algebra we see that this is a ring with unit. Since $1 \in \mathbb{Z}_x \subset {}^tK_*(G)$ acts as the identity on the successive quotients one can show by a simple induction that it is actually the identity in the bigger ring ${}^tK_*(G)$. But then if 1 is x-torsion then every element is x-torsion:

$$x\kappa = (x \cdot 1)\kappa = 0\kappa = 0 \quad \forall \kappa \in {}^tK_*(G)\,. \tag{5.9}$$

By similar arguments one can show that ${}^tK_*(G) = \mathrm{Tor}^{RG}_*(RG/I_k, \mathbb{Z})$ as a ring.

6. Level-Rank Nonduality

For WZW models there are various level-rank dualities, see [9]. Those are distinct WZW models whose fusion rings happens to be the same, for example B_2 at level $k = 1$ and E_8 at level $k = 2$. The corresponding Verlinde algebras are

$$\begin{aligned} (B_2)_1: \quad RB_2/V_1 &= \mathbb{Z}[x_1, x_2] \Big/ \big\langle x_2^2 = 1,\ x_2 x_1 = x_1,\ x_1^2 = 1 + x_2 \big\rangle \\ (E_8)_2: \quad RE_8/V_2 &= \mathbb{Z}[y_1, \ldots, y_8] \Big/ \\ &\qquad \Big/ \big\langle y_8^2 = 1,\ y_8 y_1 = y_1,\ y_1^2 = 1 + y_8,\ y_2, \ldots, y_7 \big\rangle \end{aligned} \tag{6.1a}$$

where the nontrivial generators

$$\dim_{\mathbb{C}}(x_1) = 4 \qquad \dim_{\mathbb{C}}(x_2) = 5 \qquad \dim_{\mathbb{C}}(y_1) = 248 \qquad \dim_{\mathbb{C}}(y_8) = 3875 \tag{6.2}$$

correspond to the outermost nodes in the Dynkin diagram.

Now the two Verlinde algebras eq. (6.1a), (6.1a) are obviously isomorphic algebras. These algebras appear in the computation of the twisted K-theory for

$$B_2 \quad \text{with} \quad t_{B_2} = 1 + \check{h}(B_2) = 4\,, \qquad E_8 \quad \text{with} \quad t_{E_8} = 2 + \check{h}(E_8) = 32\,. \tag{6.3}$$

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From table 1 we readily find

$$ {}^{t}K\,(B_2) \simeq \mathbb{Z}_2 \neq \mathbb{Z}_2^{128} \simeq {}^{t}K\,(E_8)\,. \tag{6.4}$$

Of course this should not be too much of a surprise, as the computation of the twisted K-theory depends on the RB_2/RE_8 module structure of the Verlinde algebra. There is no reason why level-rank duality should hold at the K-theory level.

This is in contrast to the supersymmetric Kazama-Suzuki models [19] where level-rank duality is believed to be an exact duality.

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